

ANALISI COMPLESSA
Corsi di Laurea in Matematica
Elementary functions

• **The exponential function**

For every $z \in \mathbb{C}$ define

$$e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!}.$$

The exponential is an entire function such that $D(e^z) = e^z$.

Moreover:

- $e^{z+w} = e^z e^w$ for every $z, w \in \mathbb{C}$;
- e^z is $2\pi i$ -periodic and not continuous at ∞ ;
- If $z = x + iy$, then $e^z = e^x(\cos y + i \sin y)$;
- $|e^z| = e^x$ and $\arg(e^z) = y + 2k\pi$, $k \in \mathbb{Z}$;
- for every nonzero complex number $\alpha = \rho e^{i\varphi}$ the equation $e^z = \alpha$ has the infinite solutions $z_k = \ln \rho + i(\varphi + 2k\pi)$, $k \in \mathbb{Z}$.

• **The trigonometric functions**

For every $z \in \mathbb{C}$ define

$$\sin z = \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad \cos z = \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

They are both entire functions such that $D(\sin z) = \cos z$ and $D(\cos z) = -\sin z$.

Moreover:

- (Euler formula) $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$, $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$;
- They are 2π -periodic and not continuous at ∞ ;
- $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$;
- For every $z, w \in \mathbb{C}$,

$$\begin{aligned} \sin^2 z + \cos^2 z &= 1, \\ \sin(z+w) &= \sin z \cos w + \cos z \sin w, \\ \cos(z+w) &= \cos z \cos w - \sin z \sin w; \end{aligned}$$

- If $z = x + iy$, then

$$\begin{aligned} \sin z &= \sin x \cosh y + i \cos x \sinh y, \\ \cos z &= \cos x \cosh y - i \sin x \sinh y; \end{aligned}$$

- $|\sin z|^2 = \sin^2 x + \sinh^2 y$ and $|\cos z|^2 = \cos^2 x + \sinh^2 y$;
- Both functions take every complex value infinitely many times.

• **The hyperbolic functions**

For every $z \in \mathbb{C}$ define

$$\sinh z = \sum_{n=0}^{+\infty} \frac{z^{2n+1}}{(2n+1)!}, \quad \cosh z = \sum_{n=0}^{+\infty} \frac{z^{2n}}{(2n)!}.$$

They are both entire functions such that $D(\sinh z) = \cosh z$ and $D(\cosh z) = \sinh z$.

Moreover:

- $\sinh z = \frac{1}{2}(e^z - e^{-z})$, $\cosh z = \frac{1}{2}(e^z + e^{-z})$;
- They are $2\pi i$ -periodic and not continuous at ∞ ;
- $\sinh(-z) = -\sinh z$ and $\cosh(-z) = \cosh z$;
- For every $z, w \in \mathbb{C}$,

$$\cosh^2 z - \sinh^2 z = 1,$$

$$\sinh(z+w) = \sinh z \cosh w + \cosh w \sinh z,$$

$$\cosh(z+w) = \cosh z \cosh w + \sinh w \sinh z;$$

- If $z = x + iy$, then

$$\sinh z = \sinh x \cos y + i \cosh x \sin y,$$

$$\cosh z = \cosh x \cos y + i \sinh x \sin y;$$

- $|\sinh z|^2 = \sinh^2 x + \sin^2 y$ and $|\cosh z|^2 = \cosh^2 x + \cos^2 y$;
- Both functions take every complex value infinitely many times;
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$$\cosh(iz) = \cos z, \quad \sinh(iz) = -i \sin z.$$